Convergence of the perimeter of an inscribed regular polygon

Kévin BRAULT

18/03/2025

Consider a n-sided regular polygon inscribed in a circle of radius r. The **summits of the polygon lie on the circle**, and all **sides of the polygon are of equal length**. The angle at the center of a regular polygon is the angle formed by two consecutive rays of the circle passing through the vertices of the polygon. **Since the polygon is regular, all these angles at the center are equal**.

— **Full circle circumference** : The complete circumference of a circle is 2π radians.
— **Number of sides** : The polygon has n sides.
— **Circumference division** : Each side of the polygon subtends an angle at the center that is a fraction of the total circumference. Since the polygon has n sides, the circumference is divided into n equal parts.

Therefore, the centrale angle subtended by a side of the polygon is :

$$\text{Central angle} = \frac{2\pi}{n} \text{ radians}$$

Example : For $n = 6$ (hexagon), each side subtends an angle at the center of : $\frac{2\pi}{6} = \frac{\pi}{3}$ radians

Thanks to the inscribed angle theorem , we know that :

$$\text{Base angle} = \frac{\text{Central angle}}{2} = \frac{\pi}{n}$$

Let's take the isosceles triangle OAB where :
— O is the center of the circle.
— A and B are two consecutive vertices of the polygon.
— OA and OB are the radii of the circle, each of length r.
— AB is a side of the polygon, of length s.

We have proved that :
— The centrale angle $\angle AOB = \frac{2\pi}{n}$
— And that every angle at the base $\angle OAB$ et $\angle OBA$ is $\frac{\pi}{n}$

To find the length of side s, we can use trigonometry in the OAB triangle; the definition of sine is : $\sin(\frac{\pi}{n}) = \frac{\text{opposed}}{\text{hypoténuse}}$

And as :
- The hypotenuse is the radius OA, which is r.
- The opposite side is half the side AB, which is $\frac{s}{2}$.

Then we have :

$$\sin\left(\frac{\pi}{n}\right) = \frac{s/2}{r}$$
$$\frac{s}{2} = r\sin\left(\frac{\pi}{n}\right)$$

Thus, the length of a side s of the regular polygon inscribed in a circle of radius r is given by :

$$s = 2r\sin\left(\frac{\pi}{n}\right)$$

We now have all we need to demonstrate the value of the perimeter of a polygon.

The perimeter P of the polygon is the sum of the lengths of all its sides. Since the polygon has n sides, each of length s, the perimeter is :

$$P = n \cdot s = n \cdot 2r\sin\left(\frac{\pi}{n}\right)$$

The basic question was to find out the value of the perimeter P of the inscribed regular polygon when the number of its sides n tends towards infinity.

$$\lim_{n\to\infty} P = 2\pi r$$

We know that for small angles $\sin x = x$ and when n is very large, $\frac{\pi}{n}$ is small. We can therefore use this approximation :

$$\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$$

Substituting this approximation into the perimeter expression, we obtain :

$$P \approx n \cdot 2r \cdot \frac{\pi}{n} = 2\pi r$$

Thus, when n tends to infinity, the perimeter P of the regular polygon inscribed in the circle tends to $2\pi r$, which is the circumference of the circle :

$$\lim_{n\to\infty} P = 2\pi r$$